Ambiguity and Risk in Global Games*

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June 2009

Abstract

This paper introduces *global games with multiple priors*, in which players are ambiguity averse having maxmin expected utility preferences, and studies how the distinction between risk and ambiguity matters in a global game analysis. It proposes a tractable procedure to obtain a unique equilibrium and to conduct comparative statics. As applications, global game models of currency attacks and bank runs are considered. In the model of currency attacks, news of *less* ambiguous quality can lead to the currency collapse. In the model of bank runs, news of *more* ambiguous quality can lead to the bank failure.

JEL classification numbers: C72, D81, D82.
Key words: ambiguity; bank run; currency attack; global game; higher order ambiguous beliefs; multiple priors; uniqueness.

*I thank seminar participants for valuable discussions at GRIPS, Hokkaido University, Kobe University, and Nanzan University. I acknowledge financial support by MEXT, Grant-in-Aid for Scientific Research.

1 Introduction

Incomplete information games typically assume that players assign common probabilities to all uncertain events. This immediately implies that there is no distinction between risky situations, where players know the probabilities of all events, and ambiguous situations, where they may have little confidence regarding the true probabilities. In single person decision making, on the other hand, it is well known that this distinction is behaviorally meaningful. The Ellsberg Paradox (Ellsberg, 1961) and related experimental findings[1] demonstrate that a decision maker is ambiguity averse: he prefers a risky bet (a bet on events with known probabilities) to an ambiguous bet (a bet on events with unknown probabilities), and his preferences cannot be be rationalized by any probabilistic belief. Gilboa and Shcmeidler's maximin expected utility (Gilboa and Schmeidler, 1989) and Shcmeidler's Choquet expected utility (Schmeidler, 1989) are classic models of ambiguity aversion, in which decision maker's ambiguous belief is represented by multiple priors.[2]

If the distinction between risk and ambiguity is meaningful in single person decision making, it must also be meaningful in a strategic environment with incomplete information, which involves the following delicate issue of higher order ambiguous beliefs. Consider players who have ambiguous beliefs about the state of nature. Players' rational behavior depends not only upon ambiguous beliefs about the state but also ambiguous beliefs about other players' ambiguous beliefs, ambiguous beliefs about other players' ambiguous beliefs about other players' ambiguous beliefs, and so on.

General models with higher order ambiguous beliefs are proposed and studied by Epstein and Wang (1996), Ahn (2007), and Kajii and Ui (2005, 2009). Epstein and Wang (1996) and Ahn (2007) introduce type spaces analogous to Mertens and Zamir (1985): Epstein and Wang (1996) construct type spaces consisting of hierarchy of preferences, which include maxmin expected utility preferences, and Ahn (2007) constructs type spaces consisting of hierarchy of multiple beliefs. Kajii and Ui (2005) introduce incomplete information games with multiple priors, which are incomplete information games in which players have maxmin expected utility preferences. This class of incomplete information games includes sealed bid auctions with independent private values under ambiguity studied by Salo and Weber (1995) and Lo (1998). Kajii and Ui (2005) propose two equilibrium concepts and establish their existence. In standard incomplete information games (Harsanyi, 1967–1968), a common prior is usually assumed. Thus, a common set of priors seems a natural assumption in the multiple priors models. Kajii and Ui (2009) explore the implication of this assumption by studying the agreement theorem (Aumann, 1976) and the no trade theorem (Milgrom and Stokey, 1982) in the multiple priors models following Morris (1994) and Feinberg (2000) on the common prior assumption.

In spite of these general models, few attempts have been made to study the roles of

[1] See, for example, the survey of Camerer and Weber (1992).

[2] It is well known that the Choquet expected utility with a convex capacity coincides with the maxmin expected utility with the multiple priors given by the core of the convex capacity.

higher order ambiguous beliefs in economic settings.

The purpose of this paper is to study how higher order ambiguous beliefs have an influence on strategic interactions under incomplete information. For this purpose, we focus on global games introduced by Carlsson and van Damme (1993) (henceforth, CvD). A global game is an incomplete information game in which each player observes the state of nature with a small amount of noise, where multiple equilibria would be allowed without any noise. Common knowledge of the prior generates a hierarchy of beliefs which results in a unique equilibrium if the noise is small enough. It is for the following two reasons that we restrict attention to global games. First, global games represent strategic environments with incomplete information that are rich enough to capture the important role of higher order beliefs in economic settings, but simple enough to allow tractable analysis, as Morris and Shin (2002) note. Second, global games have important practical applications in the study of crises in financial markets such as currency crises (Morris and Shin, 1998) and bank runs (Goldstein and Pauzner, 2005).

In this paper, we introduce *global games with multiple priors*, incorporating maxmin expected utility preferences into binary action global games. Each player observes the state of nature with a small amount of noise, but he has little confidence regarding the true probability distribution. More specifically, each player has a common *set* of priors and evaluates his action in terms of the minimum of interim expected payoffs, where the minimum is taken over the set of priors. An equilibrium is defined as a strategy profile in which each player chooses an action maximizing the minimum of interim expected payoffs given the opponents' strategies. If a set of priors is a singleton, then a global game with multiple priors is reduced to CvD's global game. Our formulation conforms to the formulations of Epstein and Wang (1996), Ahn (2007), and Kajii and Ui (2005).

We emphasize that the best response correspondence of a global game with multiple priors cannot be described by that of a global game with a single prior. This is because, in a global game with multiple priors, players may use different priors to evaluate different actions observing different private signals. To illustrate such a case, imagine that a player does not exactly know how precise his private signal is, but knows that its precision (the inverse of its variance) is contained in some interval. This player takes a worst case assessment of the precision, and the worst case depends upon an action to evaluate and a private signal to observe. When a private signal conveys good news for some action, he evaluates this action as if this private signal is least precise, and when a private signal conveys bad news for some action, he evaluates this action as if this private signal is most precise.

In our main results, we show that a global game with multiple priors has a unique equilibrium if the following condition is satisfied. Given a binary action global game with multiple priors, consider a fictitious global game with the same payoff function and some fixed pair of priors such that players evaluate each action by using each single prior respectively. We show that if a fictitious global game has a unique equilibrium for any pair of priors included in the set of priors, then the global game with multiple priors also has a unique equilibrium, which survives iterated deletion of strictly interim-dominated strategies. Furthermore, we show that the unique equilibrium is obtained by, roughly stated, the "maxmin" of the equilibria of the fictitious global games, where the "maxmin" is taken with respect to a pair of priors over the set of priors. This result also gives us a tractable procedure to obtain the unique equilibrium.

We study two applications using our procedure. One application is a model of currency attacks. In the global game model of self-fulfilling currency attacks studied by Morris and Shin (1998), a continuum of speculators must decide whether to attack a fixed exchange rate regime by selling the currency short, and the currency collapses in the unique equilibrium if and only if the state of fundamentals is less than some threshold. We incorporate maxmin expected utility preferences into the model of Morris and Shin (1998), assuming that speculators does not know exactly the quality of their private signals about the state of fundamentals. We show that even when the state of fundamentals is less than the threshold (i.e., the currency collapses if speculators know the quality of their private signals), the currency does not collapse if the quality of information is sufficiently ambiguous. This implies that the arrival of news with *less* ambiguous quality can lead to the collapse.

The other application is a model of bank runs. In the global game model of bank runs studied by Goldstein and Pauzner (2005), a continuum of depositors must decide whether to withdraw money from a bank or not, and the bank fails (withdrawal of the deposit is greater than the cash reserve) in the unique equilibrium if and only if the state of fundamentals is less than some threshold. We incorporate maxmin expected utility preferences into the model of Goldstein and Pauzner (2005), assuming that depositors does not know exactly the quality of their private signals about the state of fundamentals. We show that even when the state of fundamentals is greater than the threshold (i.e., the bank does not fail if depositors know the quality of their private signals), the bank fails if the quality of information is sufficiently ambiguous. This implies that the arrival of news with *more* ambiguous quality can lead to the failure.

The organization of the paper is as follows. The next section starts with an example. We demonstrate how a hierarchy of ambiguous beliefs results in a unique equilibrium and explain the different roles of risk and ambiguity. Section 3 introduces a general model and Section 4 provides a sufficient condition for the uniqueness of equilibrium together with a procedure to analyze our model. Section 5 is devoted to the applications.

2 A linear example

As an illustrative example, we incorporate maxmin expected utility preferences into the linear-normal global game introduced by CvD and elaborated by Morris and Shin (2001, 2002). There is a continuum of players,[3] each of whom has to choose an action $a \in \{0,1\}$. Action 0 is a safe action yielding a constant payoff zero. Action 1 is a risky action yielding a payoff $\theta + l - 1$, where $\theta \in \mathbb{R}$ is a randomly drawn state and $l \in [0,1]$ is the proportion of the population taking action 1. To grasp this payoff structure, assume that θ is common knowledge. If $\theta > 1$, action 1 is a strictly dominant action. If $\theta < 0$, action 0 is a strictly dominant action. In these cases, a Nash equilibrium is unique. The multiplicity case arises if $0 < \theta < 1$, in which there are two symmetric Nash equilibria (all to choose action 0 and all to choose action 1) and a non-symmetric Nash equilibrium (some to choose action 1 and others to choose action 1).

We assume that players do not exactly observe θ. The state θ is normally distributed with mean y and precision α (i.e., variance α^{-1}), and player i observes a private signal $x_i = \theta + \varepsilon_i$, where ε_i is normally distributed with mean 0 and precision β (i.e., variance β^{-1}), and ε_i, ε_j, and θ are mutually independent for $i \neq j$. Let $\xi \equiv (\alpha, \beta, y)$ be the vector of parameters specifying the probability distribution. We denote the conditional expectation operator of player i given his private signal x_i by $E_\xi[\cdot|x_i]$.

A strategy is a measurable function $s : \mathbb{R} \to \{0,1\}$ specifying an action for each possible private signal. Note that, when a player observes a private signal x and all the opponents follow a strategy s, the conditional expected proportion of the population choosing action 1 is given by $E_\xi[s(x')|x]$, where x' denotes the opponent's private signal. Thus, the conditional expected payoff to action 1 is given by $E_\xi[\theta|x] + E_\xi[s(x')|x] - 1$.

A natural kind of strategy we consider is one where a player chooses action 1 only if he observes a private signal above some cutoff point $\kappa \in \mathbb{R}$:

$$s(x|\kappa) = \begin{cases} 1 & \text{if } x > \kappa, \\ 0 & \text{if } x \leq \kappa. \end{cases}$$

We refer to this strategy $s(\cdot|\kappa) : \mathbb{R} \to \{0,1\}$ as the switching strategy with cutoff κ.

If each player knows the true value of ξ, the above setup is reduced to CvD's model. But we do not assume so. Instead, we assume that each player knows that ξ is contained in a compact set $\Xi \subsetneq \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}$ and has the corresponding maxmin expected utility preferences axiomatized by Gilboa and Schmeidler (1989). That is, each player with each private signal has a set of conditional probability distributions indexed by $\xi \in \Xi$, and evaluates his action in terms of the minimum expected payoff to the action, where the minimum is taken over Ξ.[4] For example, consider a player who observes a private signal x

[3] We consider a continuum of players because many applications of global games assume it, including Morris and Shin (1998) and Goldstein and Pauzner (2005). It is straightforward to translate our results for a symmetric two player case as in Carlsson and van Damme (1993).

[4] In this description, we define the incomplete information game in terms of interim beliefs. We can define it in terms of ex ante beliefs and prior by prior updating. That is, each player updates each prior in his set of priors to obtain the set of posteriors. This prior-by-prior updating rule for multiple priors is called the full Bayesian updating rule. For papers suggesting, deriving, or characterizing the full Bayesian updating rule in various settings, see Fagin and Halpern (1990), Wasserman and Kadane (1990),

and thinks that all the opponents follow a strategy s. Then, he prefers action 1 to action 0 if and only if

$$\min_{\xi\in\Xi}\Big(E_\xi[\theta|x]+E_\xi[s(x')|x]-1\Big)\geq 0.$$

Given the preferences, an equilibrium is defined in the standard way. A strategy profile in which all players choose the same strategy s is an equilibrium if $s(x)=1$ implies $\min_{\xi\in\Xi}(E_\xi[\theta|x]+E_\xi[s(x')|x]-1)\geq 0$ and $s(x)=0$ implies $\min_{\xi\in\Xi}(E_\xi[\theta|x]+E_\xi[s(x')|x]-1)\leq 0$.

Let us restrict attention to switching strategies and consider player's best response when all the opponents follow $s(\cdot|\kappa)$. We write

$$v_1(x,\kappa|\xi)\equiv E_\xi[\theta|x]+E_\xi[s(x'|\kappa)|x]-1, \tag{1}$$

which is the expected payoff to action 1 with respect to ξ when a player observes a private signal x and thinks that all the opponents follow $s(\cdot|\kappa)$. By the standard properties of a multivariate normal distribution, the conditional joint probability distribution of the state θ and any other player's private signal x' conditional on x is normal with the following mean vector and covariance matrix:

$$\mu_\xi(x)\equiv\begin{bmatrix}(\alpha y+\beta x)/(\alpha+\beta)\\(\alpha y+\beta x)/(\alpha+\beta)\end{bmatrix},\ \Sigma_\xi(x)\equiv\begin{bmatrix}1/(\alpha+\beta) & 1/(\alpha+\beta)\\1/(\alpha+\beta) & (\alpha+2\beta)/(\beta(\alpha+\beta))\end{bmatrix}.$$

Using this, we have

$$E_\xi[s(x'|\kappa)|x]=\mathrm{Prob}_\xi(x'>\kappa|x)=1-\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{(\alpha+2\beta)}}\left(\kappa-\frac{\alpha y+\beta x}{\alpha+\beta}\right)\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. Therefore,

$$v_1(x,\kappa|\xi)=\frac{\alpha y+\beta x}{\alpha+\beta}-\Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{(\alpha+2\beta)}}\left(\kappa-\frac{\alpha y+\beta x}{\alpha+\beta}\right)\right). \tag{2}$$

Since $v_1(x,\kappa|\xi)$ is continuous in (x,κ), strictly increasing in x, and strictly decreasing in κ, so is $\min_{\xi\in\Xi}v_1(x,\kappa|\xi)$. Hence, for each κ, there exists a unique value $b(\kappa)$ such that

$$\min_{\xi\in\Xi}v_1(b(\kappa),\kappa|\xi)=0,$$

which satisfies $\min_{\xi\in\Xi}v_1(x,\kappa|\xi)\geq 0$ for $x\geq b(\kappa)$ and $\min_{\xi\in\Xi}v_1(x,\kappa|\xi)\leq 0$ for $x\leq b(\kappa)$. This implies that, when a player thinks that all the opponents follow $s(\cdot|\kappa)$, his best response is $s(\cdot|b(\kappa))$. Since an equilibrium is a fixed point of the best-response correspondence, a strategy profile in which all players choose $s(\cdot|\kappa)$ is an equilibrium if

Jaffray (1992), Sarin and Wakker (1998), Pires (2002), Epstein and Schneider (2003), Wang (2003), and Siniscalchi (2006) among others.

and only if $\kappa = b(\kappa)$ holds, or equivalently, $\min_{\xi \in \Xi} v_1(\kappa, \kappa|\xi) = 0$. We call this equilibrium a switching equilibrium with cutoff κ.

Let $\underline{x} = \min_{(\alpha,\beta,y)\in\Xi} -\alpha y/\beta$ and $\overline{x} = \max_{(\alpha,\beta,y)\in\Xi}(\alpha + \beta - \alpha y)/\beta$. Then, it can be verified that action 0 is a dominant action when observing a private signal less than $\underline{x}$, and that action 1 is a dominant action when observing a private signal greater than $\overline{x}$. That is, a rational player chooses a strategy s satisfying

$$s(x) = \begin{cases} 1 & \text{if } x > \overline{x}, \\ 0 & \text{if } x < \underline{x}. \end{cases}$$

If a player knows that all the opponents choose action 0 if they observe private signals less than $\underline{x}$, his best response is to choose action 0 if his signal is less than $b(\underline{x})$. Similarly, if a player knows that all the opponents choose action 1 if they observe private signals greater than $\overline{x}$, his best response is to choose action 1 if his signal is greater than $b(\overline{x})$. Therefore, a rational player knowing that all the opponents are also rational chooses a strategy s satisfying

$$s(x) = \begin{cases} 1 & \text{if } x > b(\overline{x}), \\ 0 & \text{if } x < b(\underline{x}). \end{cases}$$

Repeating this discussion, we can show that if a strategy $s : \mathbb{R} \to \{0, 1\}$ survives n rounds of iterated deletion of strictly interim-dominated strategies, then it holds that

$$s(x) = \begin{cases} 1 & \text{if } x > b^n(\overline{x}), \\ 0 & \text{if } x < b^n(\underline{x}) \end{cases} \tag{3}$$

for all $n \geq 0$, where $b^0(\kappa) = \kappa$ and $b^{n+1}(\kappa) = b(b^n(\kappa))$. Now, suppose that there exists a unique value $\kappa^* \in \mathbb{R}$ such that $\kappa^* = b(\kappa^*)$. Then, not only the switching equilibrium is unique, but also the switching strategy survives iteration deletion of interim-dominated strategies by (3) because it holds that $\lim_{n\to\infty} b^n(\overline{x}) = \lim_{n\to\infty} b^n(\underline{x}) = \kappa^*$.

In the rest of this section, we consider the following specific class of Ξ. Assume that players know α and y, but they do not exactly know β; that is, the quality of information is ambiguous. More formally, let

$$\Xi = \{\xi = (\alpha, \beta, y) : \beta \in [\underline{\beta}, \overline{\beta}]\},$$

where $0 < y < 1$.[5] We write $\underline{\xi} = (\alpha, \underline{\beta}, y)$ and $\overline{\xi} = (\alpha, \overline{\beta}, y)$. We emphasize that players' preferences cannot be described by a single prior model. For example, we can show that $\overline{\xi} = \arg\min_{\xi\in\Xi} v_1(x, \kappa|\xi)$ for sufficiently small x and $\underline{\xi} = \arg\min_{\xi\in\Xi} v_1(x, \kappa|\xi)$ for sufficiently large x.[6] In other words, when receiving very bad news for action 1, a player evaluates this action as if the news is most precise, and when receiving very good news

[5]This implies that there would exist multiple equilibria if the noise were vanished.

[6]On the other hand, if $\Xi = \{\xi = (\alpha, \beta, y) : y \in [\underline{y}, \overline{y}]\}$, then players always use $\underline{y}$ in evaluating action 1. In this case, players' preferences are reduced to those in a single prior model.

for action 1, he evaluates this action as if the news is least precise.[7] This is because the following term in the right-hand side of (1) dominates when x is sufficiently small or large:

$$\min_{\xi\in\Xi} E_\xi[\theta|x] = \min_{\beta\in[\underline{\beta},\overline{\beta}]} \frac{\alpha y+\beta x}{\alpha+\beta} = \begin{cases} \dfrac{\alpha y+\overline{\beta}x}{\alpha+\overline{\beta}} & \text{if } x<y, \\ \dfrac{\alpha y+\underline{\beta}x}{\alpha+\underline{\beta}} & \text{if } x>y. \end{cases}$$

Furthermore, it can also be verified that $\overline{\xi} = \arg\min_{\xi\in\Xi} v_1(b(\kappa),\kappa|\xi)$ for sufficiently small κ and $\underline{\xi} = \arg\min_{\xi\in\Xi} v_1(b(\kappa),\kappa|\xi)$ for sufficiently large κ. This implies that, when all the opponents follow a switching strategy with a very small cutoff, the best response's cutoff is determined by the minimum precision, and when all the opponents follow a switching strategy with a very large cutoff, it is determined by the maximum precision. Therefore, the best response correspondence depends upon the multiplicity of priors.

Let us study under what condition the equilibrium is unique. Since

$$v_1(\kappa,\kappa|\xi) = \frac{\alpha y+\beta\kappa}{\alpha+\beta} - \Phi\left(\sqrt{\frac{\alpha^2\beta}{(\alpha+2\beta)(\alpha+\beta)}}(\kappa-y)\right), \tag{4}$$

we have

$$\begin{aligned} \frac{\partial}{\partial\kappa}v_1(\kappa,\kappa|\xi) &= \frac{\beta}{\alpha+\beta} - \sqrt{\frac{\alpha^2\beta}{(\alpha+2\beta)(\alpha+\beta)}}\phi\left(\sqrt{\frac{\alpha^2\beta}{(\alpha+2\beta)(\alpha+\beta)}}(\kappa-y)\right) \\ &\geq \frac{\beta}{\alpha+\beta} - \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\alpha^2\beta}{(\alpha+2\beta)(\alpha+\beta)}} \\ &= \frac{\beta}{\sqrt{2\pi}(\alpha+\beta)}\left(\sqrt{2\pi} - \sqrt{\frac{\alpha^2(\alpha+\beta)}{\beta(\alpha+2\beta)}}\right), \end{aligned}$$

where ϕ is the probability density function of the standard normal distribution. Thus, if

$$\max_{\beta\in[\underline{\beta},\overline{\beta}]} \frac{\alpha^2(\alpha+\beta)}{\beta(\alpha+2\beta)} = \frac{\alpha^2(\alpha+\underline{\beta})}{\underline{\beta}(\alpha+2\underline{\beta})} < 2\pi, \tag{5}$$

then $v_1(\kappa,\kappa|\xi)$ is strictly increasing in κ for each $\xi\in\Xi$, and so is $\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi)$. This implies the following lemma.

Lemma 1 *If* (5) *holds, then there exists a unique value* κ *solving* $\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi) = 0$. *The switching strategy with cutoff* κ *is the essentially unique strategy surviving iterated deletion of strictly interim-dominated strategies.*

This lemma implies that if the minimum precision $\underline{\beta}$ is large enough, then there exists a unique equilibrium. As in the standard global games, the "essential" qualification arises because either action may be chosen when the private signal is equal to the cutoff κ.

[7] Epstein and Schneider (2006) study the implication of this type of decision making in asset pricing theory.

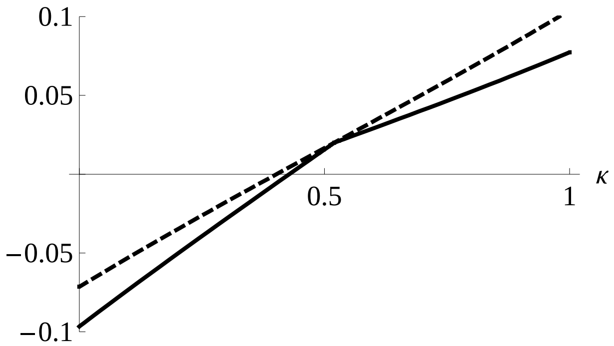

Figure 1: $\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi)$ with ambiguous information quality

Figure 1 plots $\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi)$ where $\alpha=3$, $y=0.52$, $[\underline{\beta},\overline{\beta}]=[2,3]$ for the solid line, $\underline{\beta}=\overline{\beta}=2.5$ for the dotted line, and

$$\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi)=\begin{cases}\dfrac{\alpha y+\overline{\beta}\kappa}{\alpha+\overline{\beta}}-\Phi\left(\sqrt{\dfrac{\alpha^2\overline{\beta}}{(\alpha+2\overline{\beta})(\alpha+\overline{\beta})}}(\kappa-y)\right) & \text{if } \kappa\le y,\\ \dfrac{\alpha y+\underline{\beta}\kappa}{\alpha+\underline{\beta}}-\Phi\left(\sqrt{\dfrac{\alpha^2\underline{\beta}}{(\alpha+2\underline{\beta})(\alpha+\underline{\beta})}}(\kappa-y)\right) & \text{if } \kappa> y.\end{cases}$$

In the dotted line, there is no ambiguity, but in the solid line, players receive private signals with ambiguous quality. The cutoff κ with $\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi)=0$ is given by the κ-axis intersection. Thus, Figure 1 shows that the cutoff with news of more ambiguous quality is larger, where more players take the safe action.

On the other hand, news of lower quality does not always have a similar effect on the cutoff, as Figure 2 shows. Figure 2 plots $\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi)$ where $\alpha=3$, $y=0.52$, and $\underline{\beta}=\overline{\beta}=2$ for the solid line and $\underline{\beta}=\ \overline{\beta}=2.5$ for the dotted line. There is no ambiguity in both cases, but the quality of information in the solid line is lower than that in the dotted line. Figure 2 shows that the cutoff with news of lower quality is smaller, where more players take the risky action. The intuition behind this is as follows. Consider players observing private signals less than the cutoff in the dotted line, who choose the safe action in the equilibrium. When β becomes smaller, players put more weight on y and less weight on their private signals in calculating the conditional expectation of θ. Since the cutoff is less than y and so are the private signals, smaller β increases the conditional expectation of θ. As a result, more players choose the risky action, which implies the smaller cutoff.

It can be shown that if $y>0.5$, then news of lower quality makes more players choose the risky action, and if $y<0.5$, then it makes more players choose the safe action. Thus, the effect of risky information on the cutoff depends upon the mean of the state, whereas that of ambiguous information does not, as we discuss in Section 4. In the appendix, we argue that even if players are risk averse, the effect of risky information on the cutoff depends upon the mean of the state.

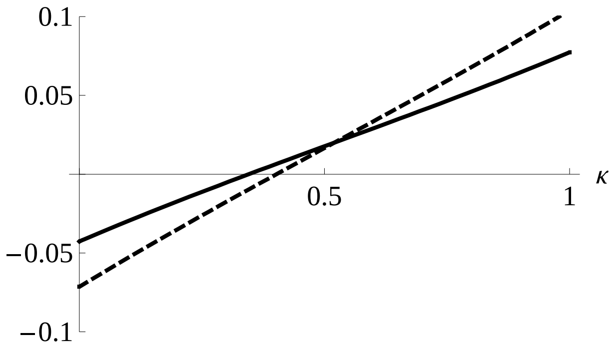

Figure 2: $\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi)$ with low information quality

3 Global games with multiple priors

We introduce a general model. There is a continuum of players, each of whom has to choose an action $a \in \{0,1\}$. All players have the same payoff function $u : \{0,1\} \times [0,1] \times \mathbb{R} \to \mathbb{R}$, where $u(a,l,\theta)$ is a payoff to action a when proportion l of the opponents choose action 1 and the true state is θ. We impose the following properties.

A1 (Action Monotonicity) $u(1,l,\theta)$ is increasing in l and $u(0,l,\theta)$ is decreasing in l.

A2 (State Monotonicity) $u(1,l,\theta)$ is increasing in θ and $u(0,l,\theta)$ is decreasing in θ.

In the standard global games, it is enough to assume that $u(1,l,\theta) - u(0,l,\theta)$ is increasing in both l and θ. We need the above stronger assumptions because, in our global games, players may use different priors when evaluating different actions.

The state θ is randomly drawn and player i observes a private signal $x_i = \theta + \varepsilon_i$, where θ, ε_i, and ε_j are mutually independent for $i \neq j$. Let $p_\xi(\theta|x)$ denote the probability density function of θ held by a player observing a private signal x, and let $q_\xi(\varepsilon)$ denote the probability density function of the noise term ε, where ξ is a parameter for the probability distribution taking a value in a compact connected set Ξ in a finite-dimensional Euclidean space. Note that, when all players choose a strategy s, the true state is θ, and the true parameter is ξ, the proportion of the population taking action 1 is calculated as $\int s(x)q_\xi(x-\theta)dx$ by the law of large numbers.

We assume that $p_\xi(\theta|x)$ and $q_\xi(\varepsilon)$ are continuous in $\xi \in \Xi$ in the weak topology. The interim belief $p_\xi(\theta|x)$ is typically the conditional probability density function: if θ has the probability density function $p_\xi(\theta)$ then $p_\xi(\theta|x) = q_\xi(x-\theta)p_\xi(\theta)/\int q_\xi(x-\theta')p_\xi(\theta')d\theta'$; if θ has the improper uniform distribution over the real line[8] then $p_\xi(\theta|x) = q_\xi(x-\theta)$. For each $\xi \in \Xi$, we impose the following properties.

A3 (Limit Dominance) There exist $\underline{\theta}, \overline{\theta}, \underline{x}, \overline{x} \in \mathbb{R}$ and $\varepsilon > 0$ satisfying the following conditions:

[8] Some applications of global games assume the improper uniform prior distribution.

- if $\theta \leq \underline{\theta}$ then $\sup_l u(1, l, \theta) - \inf_l u(0, l, \theta) \leq -\varepsilon$.
- if $\theta \geq \underline{\theta}$ and $x \leq \underline{x}$ then $p_\xi(\theta|x) \leq p_\xi(\theta|\underline{x})$ and $\lim_{x \to -\infty} p_\xi(\theta|x) = 0$.
- if $\theta \geq \overline{\theta}$ then $\inf_l u(1, l, \theta) - \sup_l u(0, l, \theta) \geq \varepsilon$.
- if $\theta \leq \overline{\theta}$ and $x \geq \overline{x}$ then $p_\xi(\theta|x) \leq p_\xi(\theta|\overline{x})$ and $\lim_{x \to \infty} p_\xi(\theta|x) = 0$.

A4 (Stochastic Dominance) If $x > x'$, then $p_\xi(\theta|x)$ first-order stochastically dominates $p_\xi(\theta|x')$.

Condition A3 says that if θ is sufficiently small (large) then action 0 (action 1) is a dominant action and a player observing a very small (large) private signal thinks that this action is highly probably a dominant action. Note that if A1 is true then $\sup_l u(1, l, \theta) - \inf_l u(0, l, \theta) = u(1, 1, \theta) - u(0, 1, \theta)$ and $\inf_l u(1, l, \theta) - \sup_l u(0, l, \theta) = u(1, 0, \theta) - u(0, 0, \theta)$. We write A3 in the above form since we will also consider the case without A1. Condition A4 says that the probability of an event in which θ is smaller than some value is decreasing in x.

We call (u, Ξ) a *global game with multiple priors*.[9] For each $\xi \in \Xi$, $(u, \{\xi\})$ is CvD's global game. In (u, Ξ), a player knows that ξ is contained in Ξ and has the corresponding maxmin expected utility preferences (Gilboa and Schmeidler, 1989). That is, each player with each private signal has a set of interim beliefs indexed by $\xi \in \Xi$, and evaluates his action in terms of the minimum expected payoff to the action, where the minimum is taken over Ξ. For example, consider a player who observes a private signal x and thinks that all the opponents follow a strategy s. Then, he prefers action a to action a' if and only if

$$\inf_{\xi \in \Xi} \int u(a, \textstyle\int s(x') q_\xi(x' - \theta) dx', \theta) p_\xi(\theta|x) d\theta$$
$$\geq \inf_{\xi \in \Xi} \int u(a', \textstyle\int s(x') q_\xi(x' - \theta) dx', \theta) p_\xi(\theta|x) d\theta. \tag{6}$$

We define a strategy profile in which all players choose s as an equilibrium if $s(x) = 1$ implies (6) with $a = 1$ and $s(x) = 0$ implies (6) with $a = 0$ for any a'.

Let $v_a(x, \kappa|\xi)$ denote the expected payoff to action a with respect to ξ when a player observes a private signal x and thinks that all the opponents follow $s(\cdot|\kappa)$:

$$v_a(x, \kappa|\xi) \equiv \int u(a, 1 - Q_\xi(\kappa - \theta), \theta) p_\xi(\theta|x) d\theta,$$

where Q_ξ is the cumulative distribution function of q_ξ. By A1, A2, and A4, $v_1(x, \kappa|\xi)$ is increasing in x and decreasing in κ; $v_0(x, \kappa|\xi)$ is decreasing in x and increasing in κ. In addition to these properties, we require the following continuity.

A5 (Continuity) $(\kappa, x, \xi) \mapsto v_a(x, \kappa|\xi)$ is continuous for each $a \in \{0, 1\}$.

[9] We assume that all players have a common prior set. Kajii and Ui (2005) and Kajii and Ui (2009) discusses some implication of the *common prior set* assumption in terms of the agreement theorem of Aumann (1976).

By A5 and compactness of Ξ, $\inf_\xi v_a(x,\kappa|\xi) = \min_\xi v_a(x,\kappa|\xi)$ and $(\kappa, x) \mapsto \min_\xi v_a(x,\kappa|\xi)$ is also continuous.

4 Main results

By the definition of an equilibrium, there exists a switching equilibrium with cutoff κ if and only if

$$\min_{\xi\in\Xi} v_1(x,\kappa|\xi) - \min_{\xi\in\Xi} v_0(x,\kappa|\xi) \begin{cases} \geq 0 & \text{if } x > \kappa, \\ \leq 0 & \text{if } x \leq \kappa. \end{cases} \tag{7}$$

This is equivalent to

$$\min_{\xi\in\Xi} v_1(\kappa,\kappa|\xi) - \min_{\xi\in\Xi} v_0(\kappa,\kappa|\xi) = 0 \tag{8}$$

because $\min_\xi v_1(x,\kappa|\xi) - \min_\xi v_0(x,\kappa|\xi)$ is continuous and increasing in x. Hence, if there exists a unique value $\kappa \in \mathbb{R}$ solving (8), a switching strategy with cutoff κ is the unique switching strategy. Furthermore, we can show that (u, Ξ) is dominance solvable by borrowing CvD's argument, as stated in the next proposition. Its proof is in the appendix.

Proposition 1 *Consider (u, Ξ) satisfying A1, A2, A3, A4, and A5. Suppose that there exists a unique value $\kappa \in \mathbb{R}$ solving (8). Then, the switching strategy with cutoff κ is the essentially unique strategy surviving iterated deletion of strictly interim-dominated strategies.*

In order to study (u, Ξ), we have to find out whether the equation (8) has a unique solution. But in general, it is not so easy to evaluate (8) directly because it includes two minimization problems for each κ.

In the following main result of this paper, we claim that we do not have to evaluate (8) directly and we can obtain the unique equilibrium more easily. Its proof is relegated to the appendix.

Proposition 2 *Consider (u, Ξ) satisfying A1, A2, A3, A4, and A5. Suppose that there exists a unique value $\kappa \in \mathbb{R}$ solving*

$$v_1(\kappa,\kappa|\xi_1) - v_0(\kappa,\kappa|\xi_0) = 0 \tag{9}$$

for each $(\xi_0, \xi_1) \in \Xi \times \Xi$ and that it is bounded over $\Xi \times \Xi$. Let $\kappa^*(\xi_0, \xi_1)$ be the unique value. Then, the switching strategy with cutoff

$$\min_{\xi_0\in\Xi} \max_{\xi_1\in\Xi} \kappa^*(\xi_0, \xi_1) \tag{10}$$

is the essentially unique strategy surviving iterated deletion of strictly interim-dominated strategies.

The equation (9) is easier to deal with than (8). By Proposition 2, we have the following procedure to study (u, Ξ).

Step 1: Check that, for each $(\xi_0, \xi_1) \in \Xi \times \Xi$, there exists a unique value $\kappa \in \mathbb{R}$ solving $v_1(\kappa, \kappa|\xi_1) - v_0(\kappa, \kappa|\xi_0) = 0$.

Step 2: Obtain the unique solution $\kappa^*(\xi_0, \xi_1)$ for each $(\xi_0, \xi_1) \in \Xi \times \Xi$.

Step 3: Evaluate $\min_{\xi_0 \in \Xi} \max_{\xi_1 \in \Xi} \kappa^*(\xi_0, \xi_1)$, which is the cutoff of the unique equilibrium.

In some applications such as the global game model of bank runs (Goldstein and Pauzner, 2005), A1 does not hold. In that case, (7) and (8) are not necessarily equivalent. Hence, we consider the following single crossing property, which is shown to guarantee the equivalence of (7) and (8).

A6: For each $\kappa \in \mathbb{R}$ and $(\xi_0, \xi_1) \in \Xi \times \Xi$, if $v_1(\kappa, \kappa|\xi_1) - v_0(\kappa, \kappa|\xi_0) = 0$, then $v_1(x, \kappa|\xi_1) - v_0(x, \kappa|\xi_0) \leq 0$ for $x \leq \kappa$ and $v_1(x, \kappa|\xi_1) - v_0(x, \kappa|\xi_0) \geq 0$ for $x \geq \kappa$.

This condition is weaker than the set of assumptions in Proposition 2 because A1, A2, and A4 imply that $v_1(x, \kappa|\xi_1) - v_0(x, \kappa|\xi_0)$ is increasing in x. As we will see later, the global game model of bank runs satisfies A6. When A1 does not hold, we can use the next proposition as far as the weak sense of uniqueness is enough. Its proof is relegated to the appendix.

Proposition 3 *Consider (u, Ξ) satisfying A2, A3, A4, A5, and A6. Suppose that there exists a unique value $\kappa \in \mathbb{R}$ solving* (9) *for each $(\xi_0, \xi_1) \in \Xi \times \Xi$ and that it is bounded over $\Xi \times \Xi$. Let $\kappa^*(\xi_0, \xi_1)$ be the unique value. Then, there exists a unique switching equilibrium and its cutoff is* (10).

If one of the actions is a safe or state-independent action, the above procedure is more tractable. We say that action $a \in \{0, 1\}$ is a *safe* action if $u(a, l, \theta)$ is independent of $(l, \theta) \in [0, 1] \times \mathbb{R}$. We say that action $a \in \{0, 1\}$ is a *state-independent* action if $u(a, l, \theta)$ is independent of $\theta \in \mathbb{R}$. Clearly, every safe action is state-independent.

We first study the case with a safe action. The next corollary asserts that, if one action is a safe action, then the uniqueness of equilibrium in $(u, \{\xi\})$ for each $\xi \in \Xi$ implies the uniqueness in (u, Ξ) and that the cutoff in (u, Ξ) is the extreme point of the cutoffs in $(u, \{\xi\})$'s.

Corollary 4 *Consider (u, Ξ) satisfying A2, A3, A4, A5, and either A1 or A6. Suppose that there exists a unique value $\kappa \in \mathbb{R}$ solving*

$$v_1(\kappa, \kappa|\xi) - v_0(\kappa, \kappa|\xi) = 0 \tag{11}$$

for each $\xi \in \Xi$ and that it is bounded over Ξ. Let $\kappa^*(\xi)$ be the unique value. If action 0 *is a safe action, then the switching equilibrium with cutoff* $\max_{\xi \in \Xi} \kappa^*(\xi)$ *is the unique equilibrium. If action* 1 *is a safe action, then the switching equilibrium with cutoff* $\min_{\xi \in \Xi} \kappa^*(\xi)$ *is the unique equilibrium.*

Proof. Let action 0 be a safe action with $u(0,l,\theta)=c\in\mathbb{R}$. By the assumption, there exists a unique value $\kappa\in\mathbb{R}$ solving

$$v_1(\kappa,\kappa|\xi)-v_0(\kappa,\kappa|\xi)=v_1(\kappa,\kappa|\xi)-c=v_1(\kappa,\kappa|\xi)-v_0(\kappa,\kappa|\xi')=0$$

for each $(\xi,\xi')\in\Xi\times\Xi$. By Proposition 2 (if A1 holds) or Proposition 3 (if A6 holds), the switching equilibrium with cutoff $\max_\xi\kappa^*(\xi)$ is the unique equilibrium. We can prove the other case with safe action 1 similarly. ■

The above corollary provides the following comparative statics result. Consider (u,Ξ_1) and (u,Ξ_2) with $\Xi_1\subsetneq\Xi_2$. We deem that players in (u,Ξ_2) confront larger ambiguity than players in (u,Ξ_1). Suppose that action 0 is a safe action. Then, the switching equilibrium with cutoff $\max_{\xi\in\Xi_n}\kappa^*(\xi)$ is the unique equilibrium of (u,Ξ_n) for $n=1,2$. Since $\max_{\xi\in\Xi_1}\kappa^*(\xi)\le\max_{\xi\in\Xi_2}\kappa^*(\xi)$, larger ambiguity results in a larger cutoff. Recall that action 0 is a safe action in the linear example in section 1. Thus, Corollary 4 explains why the cutoff increases by news of ambiguous quality as depicted in Figure 1.

Even if no action is a safe action, we can obtain a similar claim if one action is a state-independent action and θ has a uniform prior distribution.

Corollary 5 *Consider (u,Ξ) satisfying A2, A3, A4, A5, either A1 or A6, and one of the following conditions:*

- *θ has the improper uniform prior distribution over the real line.*
- *θ and ε_i have uniform prior distributions with bounded supports $[\underline{c}(\xi),\overline{c}(\xi)]$ and $[-d(\xi),d(\xi)]$ respectively such that $[\underline{\theta},\overline{\theta}]\subseteq[\underline{c}(\xi)+2d(\xi),\overline{c}(\xi)-2d(\xi)]$, where $\underline{\theta}$ and $\overline{\theta}$ are given by A3.*

Suppose that there exists a unique value $\kappa\in\mathbb{R}$ solving (11) for each $\xi\in\Xi$ and that it is bounded over Ξ. Let $\kappa^*(\xi)$ be the unique value. If action 0 is a state-independent action, then the switching equilibrium with cutoff $\max_{\xi\in\Xi}\kappa^*(\xi)$ is the unique equilibrium. If action 1 is a state-independent action, then the switching equilibrium with cutoff $\min_{\xi\in\Xi}\kappa^*(\xi)$ is the unique equilibrium.

Proof. Let action 0 be a state-independent action with $u(0,l,\theta)=f(l)$. Assume the first condition. Then, $p_\xi(\theta|x)=q_\xi(x-\theta)$, and thus

$$\begin{aligned}v_0(\kappa,\kappa|\xi)&=\int_{-\infty}^{\infty}u(0,1-Q_\xi(\kappa-\theta),\theta)q_\xi(\kappa-\theta)d\theta\\&=\int_0^1 u(0,l,\kappa-Q_\xi^{-1}(1-l))dl=\int_0^1 f(l)dl,\end{aligned}\tag{12}$$

where we use the substitution $l=1-Q_\xi(\kappa-\theta)$. By the assumption, there exists a unique value $\kappa\in\mathbb{R}$ solving

$$v_1(\kappa,\kappa|\xi)-v_0(\kappa,\kappa|\xi)=v_1(\kappa,\kappa|\xi)-\int_0^1 f(l)dl=v_1(\kappa,\kappa|\xi)-v_0(\kappa,\kappa|\xi')=0\tag{13}$$

for each $(\xi, \xi') \in \Xi \times \Xi$. By Proposition 2 (if A1 holds) or Proposition 3 (if A6 holds), the switching strategy with cutoff $\max_\xi \kappa^*(\xi)$ is the unique equilibrium.

Assume the second condition. Then,

$$p_\xi(\theta) = \begin{cases} \dfrac{1}{\overline{c}(\xi) - \underline{c}(\xi)} & \text{if } \underline{c}(\xi) \le \theta \le \overline{c}(\xi), \\ 0 & \text{otherwise,} \end{cases} \quad q_\xi(\varepsilon) \quad = \begin{cases} \dfrac{1}{2d(\xi)} & \text{if } -d(\xi) \le \varepsilon \le d(\xi), \\ 0 & \text{otherwise,} \end{cases}$$

and it can be readily shown that $p_\xi(\theta|x) = q_\xi(x - \theta)$ for each $x \in [\underline{c}(\xi) + d(\xi), \overline{c}(\xi) - d(\xi)]$. This implies that if $\kappa \in [\underline{c}(\xi)+d(\xi), \overline{c}(\xi)-d(\xi)]$ then (13) holds by (12). Hence, it is enough to show that $\kappa^*(\xi) \in [\underline{c}(\xi) + d(\xi), \overline{c}(\xi) - d(\xi)]$. Since $[\underline{\theta}, \overline{\theta}] \subseteq [\underline{c}(\xi) + 2d(\xi), \overline{c}(\xi) - 2d(\xi)]$, a player observing a private signal $x = \underline{c}(\xi) + d(\xi)$ believes that $\theta \le \underline{\theta}$ with probability one, and a player observing a private signal $x = \overline{c}(\xi) - d(\xi)$ believes that $\theta \ge \overline{\theta}$ with probability one. Thus,

$$\begin{aligned} &v_1(\underline{c}(\xi) + d(\xi), \underline{c}(\xi) + d(\xi)|\xi) - v_0(\underline{c}(\xi) + d(\xi), \underline{c}(\xi) + d(\xi)|\xi) < 0, \\ &v_1(\overline{c}(\xi) - d(\xi), \overline{c}(\xi) - d(\xi)|\xi) - v_0(\overline{c}(\xi) - d(\xi), \overline{c}(\xi) - d(\xi)|\xi) > 0. \end{aligned}$$

Since $v_1(\kappa, \kappa|\xi) - v_0(\kappa, \kappa|\xi)$ is continuous in κ by A5, $\kappa^*(\xi) \in [\underline{c}(\xi) + d(\xi), \overline{c}(\xi) - d(\xi)]$.

We can prove the other case with state-independent action 1 similarly. ■

5 Applications

5.1 Currency crises

We apply Corollary 4 to the global game model of self-fulfilling currency attacks studied by Morris and Shin (1998). Players are speculators. A continuum of players must decide whether to attack a fixed exchange rate regime by selling the currency short. Let e^* be the current value of the currency. If the monetary authority does not defend the currency, the currency will float to the shadow rate $\zeta(\theta)$, where $\theta \in \mathbb{R}$ is the state of fundamentals. The monetary authority defends the currency if the cost is not too large. The cost is increasing in the proportion of players to attack and decreasing in the state of fundamentals. Then, there exists some critical proportion of players, $a(\theta)$, increasing in θ, who must attack in order for a devaluation to occur. There is a fixed transaction cost $t > 0$ of attacking. Writing 1 for the action "not to attack" and 0 for the action "to attack," we have the following payoff function of players:

$$\begin{aligned} &u(1, l, \theta) = 0, \\ &u(0, l, \theta) = \begin{cases} e^* - \zeta(\theta) - t & \text{if } l \le 1 - a(\theta), \\ -t & \text{if } l > 1 - a(\theta). \end{cases} \end{aligned}$$

We assume that $\zeta(\theta)$ is increasing in θ with $\zeta(\theta) < e^* - t$, and that $a(\theta)$ is continuous and strictly increasing in θ with $-\infty < a^{-1}(0)$ and $a^{-1}(1) < \infty$.

If θ were common knowledge, there would be three ranges of fundamentals. If $\theta < a^{-1}(0)$, each player has a dominant action to attack. If $a^{-1}(0) \leq \theta \leq a^{-1}(1)$, then there is an equilibrium where all players attack and another equilibrium where all players do not attack. If $\theta > a^{-1}(1)$, each player has a dominant action not to attack. This tripartite division of fundamentals follows Obstfeld (1996).

Assume that the state θ has the improper uniform prior distribution and the noise term ε has a continuous probability density function $q_\xi(\varepsilon) = \sqrt{\xi}q(\sqrt{\xi}\varepsilon)$ such that $\int \varepsilon^2 q(\varepsilon)d\varepsilon = 1$ and $q(\varepsilon) = q(-\varepsilon)$, where $\xi \in \Xi \equiv [\underline{\xi}, \overline{\xi}] \subsetneq \mathbb{R}_{++}$. Then, $p_\xi(\theta|x) = q_\xi(x - \theta) = \sqrt{\xi}q(\sqrt{\xi}(x - \theta))$. Note that ξ is the precision of the noise term.

Morris and Shin (1998) study $(u, \{\xi\})$ and show that it has the unique switching strategy that survives iterated deletion of strictly interim-dominated strategies. Let $\kappa^*(\xi)$ be its cutoff, which is a unique value κ solving $v_1(\kappa, \kappa|\xi) - v_0(\kappa, \kappa|\xi) = v_0(\kappa, \kappa|\xi) = 0$.

We study (u, Ξ) assuming that $\xi \in \Xi$ is the true parameter; that is, we incorporate ambiguity about private signals' quality into the model of Morris and Shin (1998). It can be readily checked that (u, Ξ) satisfies A1, A2, A3, A4, and A5. Therefore, (u, Ξ) also has the unique switching strategy that survives iterated deletion of strictly interim-dominated strategies by Corollary 4, and its cutoff is

$$\underline{\kappa}^*(\Xi) \equiv \min_{\xi' \in \Xi} \kappa^*(\xi').$$

We study under what condition the currency collapses in the equilibrium of (u, Ξ). Recall that, when the true state is θ, the currency collapses if the proportion of players to attack exceeds $a(\theta)$. When the true state is θ, the proportion of players to attack in the switching equilibrium with cutoff $\underline{\kappa}^*(\Xi)$ is

$$\int_{-\infty}^{\underline{\kappa}^*(\Xi)} q_\xi(x - \theta)dx = \int_{-\infty}^{\underline{\kappa}^*(\Xi)} \sqrt{\xi}q(\sqrt{\xi}(x - \theta))dx = Q(\sqrt{\xi}(\underline{\kappa}^*(\Xi) - \theta)),$$

where Q is the cumulative distribution function of q. Hence, the currency collapses in (u, Ξ) if and only if

$$Q(\sqrt{\xi}(\underline{\kappa}^*(\Xi) - \theta)) \geq a(\theta). \tag{14}$$

Since the left-hand side is decreasing in θ and the right-hand side is strictly increasing in θ, there exists a unique value $\theta^*(\Xi)$ such that

$$Q(\sqrt{\xi}(\underline{\kappa}^*(\Xi) - \theta^*(\Xi))) = a(\theta^*(\Xi)).$$

As the next lemma shows, $\theta^*(\Xi)$ is the threshold of the currency collapse, and it is increasing with respect to the set inclusion.

Lemma 2 *The currency collapses in the equilibrium of (u, Ξ) if and only if $\theta \leq \theta^*(\Xi)$. If $\Xi \supseteq \Xi'$, then $\theta^*(\Xi) \leq \theta^*(\Xi')$.*

Proof. Note that $\theta \leq \theta^*(\Xi)$ is equivalent to (14) since $Q(\sqrt{\xi}(\underline{\kappa}^*(\Xi) - \theta)) - a(\theta)$ is decreasing in θ, which implies the first half of the lemma. If $\Xi \supseteq \Xi'$, then $\underline{\kappa}^*(\Xi) \leq \underline{\kappa}^*(\Xi')$ and $Q(\sqrt{\xi}(\underline{\kappa}^*(\Xi) - \theta)) \leq Q(\sqrt{\xi}(\underline{\kappa}^*(\Xi') - \theta))$, which implies that $\theta^*(\Xi) \leq \theta^*(\Xi')$. ■

If the true state θ satisfies $\theta^*(\Xi) < \theta \leq \theta^*(\{\xi\})$, then the currency collapses in $(u, \{\xi\})$, whereas it doe not in (u, Ξ). In other words, the currency may not collapse because of ambiguity even if the state of fundamentals is so bad that it would collapse without ambiguity.

To see this role of ambiguity in more detail, we consider the special case with $e^* = 1$, $0 < t < 1$, $\zeta(\theta) = 0$, and $a(\theta) = \theta$, following the formulation of Corsetti et al. (2004). We first consider $(u, \{\xi\})$ and calculate $\kappa^*(\xi)$. By Lemma 2, the currency collapses in $(u, \{\xi\})$ if and only if $\theta \leq \theta^*(\{\xi\})$. Thus, when a player receives the private signal $\kappa^*(\xi)$ in the switching equilibrium with cutoff $\kappa^*(\xi)$, his conditional probability of the currency collapse is

$$\int_{-\infty}^{\theta^*(\{\xi\})} p_\xi(\theta|\kappa^*(\xi))d\theta = \int_{-\infty}^{\theta^*(\{\xi\})} q_\xi(\kappa^*(\xi)-\theta)d\theta = 1-Q(\sqrt{\xi}(\kappa^*(\xi)-\theta^*(\{\xi\}))) = 1-\theta^*(\{\xi\}).$$

Hence, it holds that

$$\begin{aligned} v_1(\kappa^*(\xi), \kappa^*(\xi)|\xi) - v_0(\kappa^*(\xi), \kappa^*(\xi)|\xi) &= -(1-t)(1-\theta^*(\{\xi\})) + t\theta^*(\{\xi\}) \\ &= -(1-t) + \theta^*(\{\xi\}) = 0. \end{aligned}$$

Solving this, we obtain $\theta^*(\{\xi\}) = 1 - t$ and $\kappa^*(\xi) = \theta^*(\{\xi\}) + \xi^{-1/2}Q^{-1}(\theta^*(\{\xi\})) = 1 - t + \xi^{-1/2}Q^{-1}(1-t)$.

We are ready to study (u, Ξ). Since $Q^{-1}(1-t) > 0$ if $t < 1/2$ and $Q^{-1}(1-t) < 0$ if $t > 1/2$ by the symmetry of q, the equilibrium cutoff in (u, Ξ) with $\Xi = [\underline{\xi}, \overline{\xi}]$ is

$$\underline{\kappa}^*(\Xi) = \begin{cases} 1 - t + \overline{\xi}^{-1/2}Q^{-1}(1-t) & \text{if } t < 1/2, \\ 1 - t & \text{if } t = 1/2, \\ 1 - t + \underline{\xi}^{-1/2}Q^{-1}(1-t) & \text{if } t > 1/2. \end{cases} \tag{15}$$

When $t < 1/2$, $\underline{\kappa}^*(\Xi)$ is determined by the maximum precision $\overline{\xi}$, and when $t > 1/2$, it is determined by the minimum precision $\underline{\xi}$. The intuition behind this can be explained as follows. When the transaction cost is smaller, the gain obtained by the currency collapse is larger. Thus, the equilibrium cutoff must be larger. Consider a player observing a private signal larger than the equilibrium cutoff. Since a large private signal conveys good news for the state of fundamentals and bad news for attacking, the ambiguity-averse player evaluates the action to attack presuming that his private signal is the most precise. Therefore, $\underline{\kappa}^*(\Xi)$ is determined by the maximum precision $\overline{\xi}$. Similarly, when the transaction cost is larger, $\underline{\kappa}^*(\Xi)$ is determined by the minimum precision $\underline{\xi}$.

In the following lemma, we calculate the infimum of $\theta^*(\Xi)$ with respect to all possible Ξ's.

Lemma 3 *Let $c(t) \in \mathbb{R}$ be a unique value solving $Q(\sqrt{\xi}(1-t-c(t))) = c(t)$ if $t \in (0, 1/2]$ and $c(t) = 0$ if $t \in (1/2, 1)$. Then,*

$$\inf_{\Xi:\xi\in\Xi} \theta^*(\Xi) = c(t).$$

Proof. Since $Q(\sqrt{\xi}(\underline{\kappa}^*(\Xi) - \theta^*(\Xi))) = \theta^*(\Xi)$, $\underline{\kappa}^*(\Xi) \leq \underline{\kappa}^*(\Xi')$ if and only if $\theta^*(\Xi) \leq \theta^*(\Xi')$. Thus, if $\inf_{\Xi:\underline{\xi}\in\Xi} \underline{\kappa}^*(\Xi) > -\infty$, then

$$Q(\sqrt{\xi}(\inf_{\Xi:\underline{\xi}\in\Xi} \underline{\kappa}^*(\Xi) - \inf_{\Xi:\underline{\xi}\in\Xi} \theta^*(\Xi))) = \inf_{\Xi:\underline{\xi}\in\Xi} \theta^*(\Xi),$$

and if $\inf_{\Xi:\underline{\xi}\in\Xi} \underline{\kappa}^*(\Xi) = -\infty$, then $\inf_{\Xi:\underline{\xi}\in\Xi} \theta^*(\Xi) = 0$. On the other hand, (15) implies that

$$\inf_{\Xi:\underline{\xi}\in\Xi} \underline{\kappa}^*(\Xi) = \begin{cases} 1-t & \text{if } t \leq 1/2, \\ -\infty & \text{if } t > 1/2. \end{cases}$$

Combining the above, we obtain the lemma. ■

By this lemma and (15), we can summarize the relationship between the state of fundamentals and the currency's collapse under ambiguity as follows. Suppose that $\theta \leq c(t)$. Then, the currency collapses in (u, Ξ) for any Ξ. Suppose that $c(t) < \theta \leq 1 - t$. Then, if $t < 1/2$ and $\bar{\xi}$ is sufficiently large, or if $t > 1/2$ and $\underline{\xi}$ is sufficiently small, then the currency does not collapse in (u, Ξ), whereas it collapses in $(u, \{\xi\})$. Suppose that $\theta > 1 - t$. Then, the currency does not collapse in (u, Ξ) for any Ξ.

5.2 Bank runs

We apply Corollary 5 to the global game model of bank runs studied by Goldstein and Pauzner (2005), who add noises to the classic bank runs model of Diamond and Dybvig (1983). A continuum of depositors must decide whether to withdraw their money from a bank in period 1 or in period 2. The total deposits are normalized to 1. If depositors withdraw their money in period 1, they receive $r > 1$. The bank follows a sequential-service constraint: it pays r to depositors until it runs out of money. Any remaining money earns a stochastic total return in period 2, $R > 0$ with probability $\rho(\theta)$ and 0 with probability $1 - \rho(\theta)$, which is divided equally among those who chose to wait until period 2 to withdraw their money. Here, θ is the state of the economy and $\rho(\theta)$ is increasing in θ.

There are two types of depositors. With probability λ, a depositor is impatient and with probability $1-\lambda$, a depositor is patient, where it is assumed that $\lambda < r^{-1}$. Impatient depositors have consumption needs only in period 1 and always withdraw their money in period 1. Patient depositors have consumption needs only in period 2 and withdraw their money either in period 1 or in period 2.

We are concerned with the game among the proportion $1 - \lambda$ of patient depositors. Let action 0 be "withdrawal in period 1," and let action 1 be "withdrawal in period 2." Writing l for the proportion of patient depositors who choose action 1,[10] we can

[10] The proportion of depositors who withdraw in period 1 is $\lambda + (1-\lambda)(1-l) = 1 - l(1-\lambda)$. Thus, the bank runs out of money if $(1 - l(1-\lambda))r \geq 1$, i.e., $l \leq (1 - r^{-1})/(1-\lambda)$.

summarize the money payoffs as follows.

	$l \leq \frac{1-r^{-1}}{1-\lambda}$	$l > \frac{1-r^{-1}}{1-\lambda}$
action 0	$\begin{cases} r & \text{with prob. } \frac{1}{1-l(1-\lambda)} \\ 0 & \text{with prob. } 1-\frac{1}{1-l(1-\lambda)} \end{cases}$	r
action 1	0	$\begin{cases} \left(r-\frac{r-1}{l(1-\lambda)}\right)R & \text{with prob. } \rho(\theta) \\ 0 & \text{with prob. } 1-\rho(\theta) \end{cases}$

Depositors obtain utility $U(y)$ from consumption y with $U(0) = 0$, where the relative risk aversion coefficient of U is strictly greater than 1. Then, patient depositors have the following payoff function:

$$u(0,l,\theta) = \begin{cases} \frac{1}{1-l(1-\lambda)}U(r) & \text{if } l \leq \frac{1-r^{-1}}{1-\lambda}, \\ U(r) & \text{if } l > \frac{1-r^{-1}}{1-\lambda}, \end{cases}$$

$$u(1,l,\theta) = \begin{cases} 0 & \text{if } l \leq \frac{1-r^{-1}}{1-\lambda}, \\ \rho(\theta)U\left(\left(r-\frac{r-1}{l(1-\lambda)}\right)R\right) & \text{if } l > \frac{1-r^{-1}}{1-\lambda}. \end{cases}$$

Following Goldstein and Pauzner (2005), we assume that there exist $0 < \underline{\theta} < \overline{\theta} < 1$ such that if $\theta \leq \underline{\theta}$ then action 0 is a dominant action and if $\theta \geq \overline{\theta}$ then action 1 is a dominant action. Note that, if θ were common knowledge and $\underline{\theta} < \theta < \overline{\theta}$, then there are an equilibrium where all patient depositors choose action 0 and another equilibrium where all patient depositors choose action 1. In the former equilibrium, the bank fails.

The state θ has the uniform distribution over $[0,1]$ and the noise term has the uniform distribution over $[-\xi, \xi]$, where $\xi \in \Xi \equiv [\underline{\xi}, \overline{\xi}]$. We assume that

$$\underline{\xi} \leq \overline{\xi} < \min\{1/4, \underline{\theta}/2, (1-\overline{\theta})/2\},$$

which implies the second condition in Corollary 5. Then, it can be verified that $p_\xi(\theta|x) = 1/2\xi$ for $\theta \in [\xi, 1-\xi]$.

The model of Goldstein and Pauzner (2005) is $(u, \{\xi\})$. Since $u(0,l,\theta)$ is not decreasing in l, condition A1 is violated, implying that CvD's result does not apply to this game. Goldstein and Pauzner (2005) show that $(u, \{\xi\})$ has a unique equilibrium and it is a switching equilibrium. Let $\kappa^*(\xi)$ be the cutoff. In the equilibrium, the ex ante probability of the bank's failure is strictly positive, and when the bank fails, welfare of depositors is lower than what could be obtained without banks, i.e., in the autarkic allocation, where impatient agents consume one unit in period 1 and patient agents consume R units in period 2 with probability $\rho(\theta)$. By considering infinitesimally small ξ, Goldstein and Pauzner (2005) demonstrate that, in the equilibrium, the ex ante expected welfare of depositors is greater than that in the autarkic allocation when $r > 1$ is chosen optimally. In this sense, banks are shown to be viable even if banks may fail with a positive probability.

We study (u, Ξ) assuming that $\xi \in \Xi$ is the true parameter; that is, we incorporate ambiguity about private signals' quality into the model of Goldstein and Pauzner (2005).

It can be readily shown that (u, Ξ) satisfies A2, A3, A4, A5, and A6. Therefore, (u, Ξ) also has the unique switching equilibrium by Corollary 5, and its cutoff is

$$\overline{\kappa}^*(\Xi) \equiv \max_{\xi' \in \Xi} \kappa^*(\xi')$$

since action 0 is a state-independent action. Note that $\kappa^*(\xi) \leq \overline{\kappa}^*(\Xi)$. This implies that more patient depositors withdraw in period 1 and the probability of bank runs is greater under ambiguity. Especially, if $\kappa^*(\xi) + 2\xi \leq \overline{\kappa}^*(\Xi)$ and θ satisfies

$$\kappa^*(\xi) + \xi \leq \theta \leq \overline{\kappa}^*(\Xi) - \xi,$$

then all patient depositors withdraw in period 1, whereas no patient depositors would do so if there were no ambiguity.[11] In other words, the bank can fail because of ambiguity.

It should be noted that this failure can be prevented by decreasing ambiguity about the quality of information on the state of the economy. In this respect, it is important for the authorities to publish not only central tendencies but also ranges in forecasting the state of the economy. Put it differently, it is important to evaluate and publish the precision of their projection as well as increase the precision. In fact, many central banks have begun to express their views about the likely future path of the economy more openly, including both central tendencies and ranges,[12] in line with a general trend toward greater central bank transparency.

Appendix

Proof of results in section 4

We prove Proposition 1, Proposition 2, and Proposition 3. We start with the following three lemmas.

Lemma A *Consider (u, Ξ) satisfying A1, A2, A5. Then, (u, Ξ) satisfies strategic complementarities in the following sense: for $s : \mathbb{R} \to \{0, 1\}$ with $s(x|\kappa) \leq s(x)$ for all $x \in \mathbb{R}$,*

$$\min_{\xi \in \Xi} v_1(x, \kappa|\xi) - \min_{\xi \in \Xi} v_0(x, \kappa|\xi) \leq \inf_{\xi \in \Xi} \int u(a, \textstyle\int s(x')q_\xi(x' - \theta)dx', \theta)p_\xi(\theta|x)d\theta - \inf_{\xi \in \Xi} \int u(a', \textstyle\int s(x')q_\xi(x' - \theta)dx', \theta)p_\xi(\theta|x)d\theta,$$

and for $s : \mathbb{R} \to \{0, 1\}$ with $s(x|\kappa) \geq s(x)$ for all $x \in \mathbb{R}$,

$$\min_{\xi \in \Xi} v_1(x, \kappa|\xi) - \min_{\xi \in \Xi} v_0(x, \kappa|\xi) \geq \inf_{\xi \in \Xi} \int u(a, \textstyle\int s(x')q_\xi(x' - \theta)dx', \theta)p_\xi(\theta|x)d\theta - \inf_{\xi \in \Xi} \int u(a', \textstyle\int s(x')q_\xi(x' - \theta)dx', \theta)p_\xi(\theta|x)d\theta.$$

[11] It can be also shown that, if θ satisfies $\kappa^*(\xi) + (1 + \lambda - 2r^{-1})\xi/(1 - \lambda) < \theta \leq \overline{\kappa}^*(\Xi) + (1 + \lambda - 2r^{-1})\xi/(1 - \lambda)$, then the bank run out of money in (u, Ξ), whereas it does not in $(u, \{\xi\})$.

[12] The most notable example is the fan chart of Bank of England.

Proof. By A1, if $s(x|\kappa) \leq s(x)$ for all $x \in \mathbb{R}$, then

$$u(1, \int s(x'|\kappa)q_\xi(x'-\theta)dx', \theta) \leq u(1, \int s(x')q_\xi(x'-\theta)dx', \theta),$$
$$u(0, \int s(x'|\kappa)q_\xi(x'-\theta)dx', \theta) \geq u(0, \int s(x')q_\xi(x'-\theta)dx', \theta).$$

Thus, for each $(\xi_0, \xi_1) \in \Xi \times \Xi$,

$$v_1(x, \kappa|\xi_1) \leq \int u(a, \int s(x')q_\xi(x'-\theta)dx', \theta)p_{\xi_1}(\theta|x)d\theta,$$
$$v_0(x, \kappa|\xi_0) \geq \int u(a, \int s(x')q_\xi(x'-\theta)dx', \theta)p_{\xi_0}(\theta|x)d\theta,$$

and

$$v_1(x, \kappa|\xi_1) - v_0(x, \kappa|\xi_0) \leq \int u(a, \int s(x')q_\xi(x'-\theta)dx', \theta)p_{\xi_1}(\theta|x)d\theta$$
$$- \int u(a', \int s(x')q_\xi(x'-\theta)dx', \theta)p_{\xi_0}(\theta|x)d\theta,$$

Taking the infimum with respect to ξ_1 and the supremum with respect to ξ_0 in the above inequality, we obtain the first half of the lemma. We can prove the second half similarly. ■

Lemma B *Consider (u, Ξ) satisfying A3. There exist $\underline{X}, \overline{X} \in \mathbb{R}$ such that, for each $\kappa \in \mathbb{R}$, $\min_{\xi_1 \in \Xi_1} v_1(x, \kappa|\xi_1) - \min_{\xi_0 \in \Xi_0} v_0(x, \kappa|\xi_0) < 0$ for all $x \leq \underline{X}$ and $\min_{\xi_1 \in \Xi_1} v_1(x, \kappa|\xi_1) - \min_{\xi_0 \in \Xi_0} v_0(x, \kappa|\xi_0) > 0$ for all $x \geq \overline{X}$.*

Proof. Let $f_1(\theta) = \sup_l u(1, l, \theta)$ and $f_0(\theta) = \inf_l u(0, l, \theta)$. Then,

$$v_1(x, \kappa|\xi) = \int u(a, 1 - Q_\xi(\kappa - \theta), \theta)p_\xi(\theta|x)d\theta \leq \int f_1(\theta)p_\xi(\theta|x)d\theta,$$
$$v_0(x, \kappa|\xi) = \int u(a, 1 - Q_\xi(\kappa - \theta), \theta)p_\xi(\theta|x)d\theta \geq \int f_0(\theta)p_\xi(\theta|x)d\theta.$$

Let $\xi^* \in \arg\min_{\xi_0 \in \Xi} v_0(x, \kappa|\xi_0)$. Then,

$$\min_{\xi_1 \in \Xi} v_1(x, \kappa|\xi_1) - \min_{\xi_0 \in \Xi} v_0(x, \kappa|\xi_0) \leq v_1(x, \kappa|\xi^*) - v_0(x, \kappa|\xi^*).$$

Let $\underline{\theta}, \overline{\theta}, \underline{x}, \overline{x} \in \mathbb{R}$ and $\varepsilon > 0$ be given by A3. Then,

$$v_1(x, \kappa|\xi^*) - v_0(x, \kappa|\xi^*) \leq \int (f_1(\theta) - f_0(\theta))p_{\xi^*}(\theta|x)d\theta$$
$$= \int_{-\infty}^{\underline{\theta}} (f_1(\theta) - f_0(\theta))p_{\xi^*}(\theta|x)d\theta + \int_{\underline{\theta}}^{\infty} (f_1(\theta) - f_0(\theta))p_{\xi^*}(\theta|x)d\theta$$
$$\leq -\varepsilon \int_{-\infty}^{\underline{\theta}} p_{\xi^*}(\theta|x)d\theta + \int_{\underline{\theta}}^{\infty} |f_1(\theta) - f_0(\theta)|p_{\xi^*}(\theta|x)d\theta.$$

Since $|f_1(\theta) - f_0(\theta)|p_{\xi^*}(\theta|x) \leq |f_1(\theta) - f_0(\theta)|p_{\xi^*}(\theta|\underline{x})$ and $\lim_{x \to -\infty} p_{\xi^*}(\theta|x) = 0$ for $\theta \geq \overline{\theta}$, the limit of the second term is zero by the Lebesgue convergence theorem. Accordingly, $\min_{\xi_1 \in \Xi} v_1(x, \kappa|\xi_1) - \min_{\xi_0 \in \Xi} v_0(x, \kappa|\xi_0) < 0$ for sufficiently small x. The other inequality can be proved similarly. ■

Lemma C *Consider (u, Ξ) satisfying A1, A2, A4, and A5. The function $(x, \kappa) \mapsto \min_{\xi \in \Xi} v_1(x, \kappa|\xi) - \min_{\xi \in \Xi} v_0(x, \kappa|\xi)$ is continuous and increasing in x, and decreasing in κ.*

Proof. The function is continuous by A5, increasing in x by A1, A2, and A4, and decreasing in κ by A2 ■

Given the above three lemmas, we can prove Proposition 1 by the same method as that of CvD. For completeness, we provide it below.

Proof of Proposition 1. We will argue by induction that a strategy s survives n rounds of iterated deletion of strictly interim-dominated strategies if and only if

$$s(x) = \begin{cases} 0 & \text{if } x < \underline{\kappa}_n, \\ 1 & \text{if } x > \overline{\kappa}_n, \end{cases}$$

where $\underline{\kappa}_0 = -\infty$ and $\overline{\kappa}_0 = \infty$, and $\underline{\kappa}_n$ and $\overline{\kappa}_n$ are defined inductively by

$$\underline{\kappa}_{n+1} = \min\{x \in \mathbb{R} : \min_{\xi \in \Xi} v_1(x, \underline{\kappa}_n|\xi) - \min_{\xi \in \Xi} v_0(x, \underline{\kappa}_n|\xi) = 0\},$$
$$\overline{\kappa}_{n+1} = \max\{x \in \mathbb{R} : \min_{\xi \in \Xi} v_1(x, \overline{\kappa}_n|\xi) - \min_{\xi \in \Xi} v_0(x, \overline{\kappa}_n|\xi) = 0\}.$$

As an induction hypothesis, suppose that the above claim is true for n. By strategic complementarities of Lemma A, if action 1 is a best response to a strategy surviving n rounds, it must be a best response to the switching strategy with cutoff $\underline{\kappa}_n$, and $\underline{\kappa}_{n+1}$ is defined to be the lowest signal where this is the case because $\min\{x \in \mathbb{R} : \min_{\xi \in \Xi} v_1(x, \underline{\kappa}_n|\xi) - \min_{\xi \in \Xi} v_0(x, \underline{\kappa}_n|\xi)$ is increasing in x by Lemma C. Similarly, if action 0 were ever to be a best response to a strategy surviving n rounds, it must be a best response to the switching strategy with cutoff $\overline{\kappa}_n$, and $\overline{\kappa}_{n+1}$ is define to be the highest signal where this is the case.

Let $\underline{X}, \overline{X} \in \mathbb{R}$ be given in Lemma B. Then, by the above argument, $\underline{\kappa}_n < \overline{X}$ and $\overline{\kappa}_n > \underline{X}$. In addition, $\underline{\kappa}_n$ is increasing in n and $\overline{\kappa}_n$ is decreasing in n since $\underline{\kappa}_0 = -\infty < \underline{X} < \underline{\kappa}_1$, $\overline{\kappa}_0 = \infty > \overline{X} > \overline{\kappa}_1$, and $\min_{\xi \in \Xi} v_1(x, \kappa|\xi) - \min_{\xi \in \Xi} v_0(x, \kappa|\xi)$ is increasing in x and decreasing in κ by Lemma C. Thus, there exist $\underline{\kappa} = \lim_{n \to \infty} \underline{\kappa}_n$ and $\overline{\kappa} = \lim_{n \to \infty} \overline{\kappa}_n$. Since $\min_{\xi \in \Xi} v_1(x, \underline{\kappa}_n|\xi) - \min_{\xi \in \Xi} v_0(x, \underline{\kappa}_n|\xi)$ is continuous by Lemma C, the construction of $\underline{\kappa}$ and $\overline{\kappa}$ implies that

$$\min_{\xi \in \Xi} v_1(x, \underline{\kappa}|\xi) - \min_{\xi \in \Xi} v_0(x, \underline{\kappa}|\xi) = \min_{\xi \in \Xi} v_1(x, \overline{\kappa}|\xi) - \min_{\xi \in \Xi} v_0(x, \overline{\kappa}|\xi) = 0.$$

Since there exists a unique value $\kappa \in \mathbb{R}$ solving (8), $\underline{\kappa}$ and $\overline{\kappa}$ must be equal to the unique value. Therefore, the switching strategy with cutoff κ is the essentially unique strategy surviving iterated deletion of strictly interim-dominated strategies. ■

To prove Proposition 2 and Proposition 3, we use the following lemma.

Lemma D *Let* $f : \mathbb{R} \times \Xi \to \mathbb{R}$ *be a continuous function satisfying the following condition: for each* $\xi \in \Xi$*, an equation* $f(x, \xi) = 0$ *has a unique solution* $x^*(\xi)$ *and it is bounded over* Ξ*. If there exists* $\underline{X} \in \mathbb{R}$ *such that* $\min_{\xi \in \Xi} f(x, \xi) < 0$ *for all* $x \leq \underline{X}$*, then* $\max_{\xi \in \Xi} x^*(\xi)$ *is the unique solution of* $\min_{\xi \in \Xi} f(x, \xi) = 0$*. If there exists* $\overline{X} \in \mathbb{R}$ *such that* $\max_{\xi \in \Xi} f(x, \xi) > 0$ *for all* $x \geq \overline{X}$*, then* $\min_{\xi \in \Xi} x^*(\xi)$ *is the unique solution of* $\max_{\xi \in \Xi} f(x, \xi) = 0$.

Proof. Observe that $x^*(\xi)$ is continuous in ξ. To see this, suppose otherwise that there exists a convergent sequence $\{\xi_k\}$ with $\lim_{k \to \infty} \xi_k = \xi'$ such that $x^*(\xi_k)$ does not converge to $x^*(\xi')$. Since $x^*(\xi)$ is bounded over Ξ, we can choose $\{\xi_k\}$ such that $\{x^*(\xi_k)\}$ is a convergent sequence with $\lim_{k \to \infty} x^*(\xi_k) = x' \neq x^*(\xi')$. Since f is continuous, $0 = \lim_{k \to \infty} f(x^*(\xi_k), \xi_k) = f(x', \xi')$, which contradicts to the assumption that $x^*(\xi')$ is the unique solution of $f(x, \xi') = 0$. Therefore, $x^*(\xi)$ is continuous in ξ.

Next, we show that $f(x, \xi) < 0$ if $x < x^*(\xi)$ for each $\xi \in \Xi$. Seeking a contradiction, suppose otherwise that there exists $\xi' \in \Xi$ such that $f(x, \xi') > 0$ for $x < x^*(\xi')$. Since $\min_{\xi \in \Xi} f(x, \xi) < 0$ for $x \leq \underline{X}$, there exists $\xi'' \in \Xi$ such that $f(x, \xi'') < 0$ for $x < x^*(\xi'')$. Since Ξ is a connected finite dimensional space, there exists a continuous function $\phi : [0, 1] \to \Xi$ such that $\phi(0) = \xi'$ and $\phi(1) = \xi''$. Consider a continuous function $\Phi(t) = f(x^*(\phi(t)) - 1, \phi(t))$. Since $\Phi(0) = f(x^*(\phi(0)) - 1, \phi(0)) = f(x^*(\xi') - 1, \xi') > 0$ and $\Phi(1) = f(x^*(\phi(1)) - 1, \phi(1)) = f(x^*(\xi'') - 1, \xi'') < 0$, there exists t' such that $\Phi(t') = f(x^*(\phi(t')) - 1, \phi(t')) = 0$, which contradicts to the assumption that $x^*(\phi(t'))$ is the unique solution of $f(x, \phi(t')) = 0$.

Now, we show that $\max_{\xi \in \Xi} x^*(\xi)$ is the unique solution of $\min_{\xi \in \Xi} f(x, \xi) = 0$. Since $f(x, \xi) < 0$ for $x < x^*(\xi)$, $\min_{\xi \in \Xi} f(x, \xi) < 0$ for $x < \inf_{\xi \in \Xi} x^*(\xi)$ and $\min_{\xi \in \Xi} f(x, \xi) > 0$ for $x > \sup_{\xi \in \Xi} x^*(\xi)$. Since $\min_{\xi \in \Xi} f(x, \xi)$ is continuous in x, $\min_{\xi \in \Xi} f(x, \xi) = 0$ has a solution x°. Let $\xi^\circ \in \Xi$ be such that $\min_{\xi \in \Xi} f(x^\circ, \xi) = f(x^\circ, \xi^\circ)=0$. We show that $x^\circ = \max_{\xi \in \Xi} x^*(\xi)$. Seeking a contradiction, assume otherwise; that is, there exists $\xi' \in \Xi$ such that $x^\circ = x^*(\xi^\circ) < x^*(\xi')$. Then, $0 = \min_{\xi \in \Xi} f(x^\circ, \xi) \leq f(x^\circ, \xi') < 0$, which is a contradiction. Thus, we must have $x^\circ = x^*(\xi^\circ) = \sup_{\xi \in \Xi} x^*(\xi) = \max_{\xi \in \Xi} x^*(\xi)$.

By a similar argument, we can show that $\min_{\xi \in \Xi} x^*(\xi)$ is the unique solution of $\max_{\xi \in \Xi} f(x, \xi) = 0$. ■

The above lemma implies the next lemma. Proposition 2 is an immediate consequence of this lemma and Proposition 1.

Lemma E *Consider* (u, Ξ) *satisfying A2, A3, A4, and A5. Suppose that there exists a unique value* $\kappa \in \mathbb{R}$ *solving* (9) *for each* $(\xi_0, \xi_1) \in \Xi \times \Xi$ *and that it is bounded over* $\Xi \times \Xi$*. Let* $\kappa^*(\xi_0, \xi_1)$ *be the unique value. Then, there exists a unique value* $\kappa \in \mathbb{R}$ *solving* (8) *and it is given by* (10).

Proof. Let $f(\kappa, \xi_0, \xi_1) = v_1(\kappa, \kappa|\xi_1) - v_0(\kappa, \kappa|\xi_0)$. By the assumption, there exists a unique $\kappa^*(\xi_0, \xi_1)$ satisfying $f(\kappa^*(\xi_0, \xi_1), \xi_0, \xi_1) = 0$ for each $(\xi_0, \xi_1) \in \Xi \times \Xi$. By A5, $(\kappa, \xi_0, \xi_1) \mapsto$

(κ, ξ_0, ξ_1) is continuous. By A3 and Lemma B, for $\kappa < \underline{X}$,

$$\min_{\xi_1} f(\kappa, \xi_0, \xi_1) \leq \min_{\xi \in \Xi} v_1(\kappa, \kappa|\xi) - \min_{\xi \in \Xi} v_0(\kappa, \kappa|\xi) < 0.$$

By Lemma D, $\max_{\xi_1} \kappa^*(\xi_0, \xi_1)$ is the unique solution of $\min_{\xi_1} f(\kappa, \xi_0, \xi_1) = 0$.

By A5, $(\kappa, \xi_0) \mapsto \min_{\xi_1} f(\kappa, \xi_0, \xi_1)$ is continuous. By A3 and Lemma B, for $\kappa > \overline{X}$,

$$\max_{\xi_0} \min_{\xi_1} f(\kappa, \xi_0, \xi_1) = \min_{\xi \in \Xi} v_1(\kappa, \kappa|\xi) - \min_{\xi \in \Xi} v_0(\kappa, \kappa|\xi) > 0.$$

By Lemma D, $\min_{\xi_0} \max_{\xi_1} \kappa^*(\xi_0, \xi_1)$ is the unique solution of $\max_{\xi_0} \min_{\xi_1} f(\kappa, \xi_0, \xi_1) = \min_{\xi \in \Xi} v_1(\kappa, \kappa|\xi) - \min_{\xi \in \Xi} v_0(\kappa, \kappa|\xi) = 0$. ■

Proposition 3 is an immediate consequence of Lemma E and the following lemma.

Lemma F *Consider (u, Ξ) satisfying A2, A3, A4, A5, and A6. A strategy profile in which all players choose $s(\cdot|\kappa)$ is a switching equilibrium if and only if κ solves* (8)*.*

Proof. A5 implies the "only if" part. To establish the "if" part, let $\kappa \in \mathbb{R}$ solves $\min_\xi v_1(\kappa, \kappa|\xi) - \min_\xi v_0(\kappa, \kappa|\xi) = 0$. It is enough to show that

$$\min_{\xi \in \Xi} v_1(x, \kappa|\xi) - \min_{\xi \in \Xi} v_0(x, \kappa|\xi) \begin{cases} \geq 0 & \text{if } x \geq \kappa, \\ \leq 0 & \text{if } x \leq \kappa. \end{cases}$$

We first show that $\min_\xi v_1(x, \kappa|\xi) - \min_\xi v_0(x, \kappa|\xi) \leq 0$ if $x \leq \kappa$. Seeking a contradiction, suppose that there exists $x' \leq \kappa$ such that $\min_\xi v_1(x', \kappa|\xi) - \min_\xi v_0(x', \kappa|\xi) > 0$. Let $\xi_0, \xi_1, \xi_0', \xi_1' \in \Xi$ be such that $\xi_0 \in \arg\min_\xi v_0(\kappa, \kappa|\xi)$, $\xi_1 \in \arg\min_\xi v_1(\kappa, \kappa|\xi)$, $\xi_0' \in \arg\min_\xi v_0(x', \kappa|\xi)$, and $\xi_1' \in \arg\min_\xi v_1(x', \kappa|\xi)$. Then,

$$v_1(x', \kappa|\xi_1) - v_0(x', \kappa|\xi_0') \geq v_1(x', \kappa|\xi_1') - v_0(x', \kappa|\xi_0') > 0.$$

This and A6 imply that $v_1(\kappa, \kappa|\xi_1) - v_0(\kappa, \kappa|\xi_0') > 0$ since $\kappa > x'$. Thus,

$$\min_\xi v_1(\kappa, \kappa|\xi) - \min_\xi v_0(\kappa, \kappa|\xi) \geq v_1(\kappa, \kappa|\xi_1) - v_0(\kappa, \kappa|\xi_0') > 0,$$

which is a contradiction. This implies that $\min_\xi v_1(x, \kappa|\xi) - \min_\xi v_0(x, \kappa|\xi) \leq 0$ if $x \leq \kappa$. Similarly, we can show that $\min_\xi v_1(x, \kappa|\xi) - \min_\xi v_0(x, \kappa|\xi) \geq 0$ if $x \geq \kappa$. ■

Risk averse players

We show in Section 2 that news of lower quality can make more players take the risky action assuming that players are risk neutral. In this appendix, we argue that this is true even if players are risk averse.

Assume that there is no ambiguity and that payoffs are the conditional expectation of $\theta + l - 1$ minus its conditional variance multiplied by a constant. Then, the expected payoff to the risky action of a player who observes a private signal x and thinks that all the opponents follow $s(\cdot|\kappa)$ is $v_1(x,\kappa|\xi)-c/(\alpha+\beta)$, where $c \geq 0$ and $v_1(x,\kappa|\xi)$ is given by (2). By the same argument as that in Section 2, this game has a unique switching equilibrium if $\alpha^2(\alpha+\beta)/\beta(\alpha+2\beta) < 2\pi$, and the cutoff κ is a unique solution of $v_1(\kappa,\kappa|\xi)-c/(\alpha+\beta) = 0$. Let $\kappa^*(\beta)$ be the cutoff when the precision of ε_i is β. The next lemma states that if y is large enough then $\kappa^*(\beta)$ is increasing in β; that is, news of lower quality makes more players take the risky action.

Lemma G *For $\beta_1, \beta_2 > 0$, suppose that $\alpha^2(\alpha+\beta_n)/\beta_n(\alpha+2\beta_n) < 2\pi$ for $n = 1, 2$. If $y > c/\alpha + 0.5$ and $\beta_1 > \beta_2$, then $\kappa^*(\beta_1) > \kappa^*(\beta_2)$.*

Proof. Let

$$\begin{aligned} f(\kappa, y, \beta) &\equiv v_1(\kappa,\kappa|\xi) - \frac{c}{\alpha+\beta} \\ &= \frac{\alpha(y-c/\alpha)+\beta\kappa}{\alpha+\beta} - \Phi\left(\sqrt{\frac{\alpha^2\beta}{(\alpha+2\beta)(\alpha+\beta)}}(\kappa-y)\right). \end{aligned}$$

By the assumption, $f(\kappa, y, \beta_n)$ is strictly increasing in κ and $f(\kappa^*(\beta_n), y, \beta_n) = 0$ for $n = 1, 2$. Since $y > c/\alpha + 0.5$, $f(0.5, y, \beta_1) > 0$. Thus, we must have $\kappa^*(\beta_1) < 0.5 < y$. Furthermore, it can be verified that

$$\frac{\alpha y + \beta_1\kappa^*(\beta_1)}{\alpha+\beta_1} < \frac{\alpha y + \beta_2\kappa^*(\beta_1)}{\alpha+\beta_2},$$

$$\Phi\left(\sqrt{\frac{\alpha^2\beta_1}{(\alpha+2\beta_1)(\alpha+\beta_1)}}(\kappa^*(\beta_1)-y)\right) > \Phi\left(\sqrt{\frac{\alpha^2\beta_2}{(\alpha+2\beta_2)(\alpha+\beta_2)}}(\kappa^*(\beta_1)-y)\right).$$

Therefore, $f(\kappa^*(\beta_1), y, \beta_2) > f(\kappa^*(\beta_1), y, \beta_1) = 0$, and $\kappa^*(\beta_1) > \kappa^*(\beta_2)$ must follow. ■

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